

4. Entropy and Potential Temperature

One of the troubling aspects of the equation (3.4) or (3.7) is the apparent non-uniqueness of the response of substance to some fixed amount of added or subtracted heat, which results at generally arbitrary trajectories on the p - v -diagram. This is primarily because of the fact that $p dv$ or $v dp$ terms are not the true differentials as they depend on the temperature as well. Can the law be rewritten in the form that uses only true differentials? Yes, it can, and the quantity that helps us to do just that is called *entropy*.

Let's rewrite (3.7) using the equation of state (1.2)

$$\delta q = c_p dT - \frac{RT}{p} dp = c_p dT - RT \frac{dp}{p} \quad (4.1)$$

Let's now divide both sides of (4.1) by T

$$\frac{\delta q}{T} = c_p \frac{dT}{T} - R \frac{dp}{p} \quad (4.2)$$

or

$$\frac{\delta q}{T} = c_p d(\ln T) - R d(\ln p) \quad (4.3)$$

Something remarkable has just happened. Namely, the r.h.s. of (4.3), unlike (4.1), contains now only the true differentials of the natural logarithm (c_p and R are constants). That means that the left-hand-side must also be a true differential! That is, unlike integral of δq , the integral of $\delta q/T$ does not depend on path, but only on the initial and final states of the parcel! The quantity S with the differential defined as

$$dS \equiv \frac{\delta Q}{T} \quad (4.4)$$

is called the *entropy*. Now the 1st Law of Thermodynamics can be re-written as

$$\boxed{TdS = c_p dT - v dp} \quad (4.5)$$

Let's now find the way to compute the entropy of a dry air. Using (4.3)

$$dS = c_p d(\ln T) - R d(\ln p) = c_p d(\ln T) - d(\ln p^{R/c_p}) = c_p d(\ln [T p^{-R/c_p}]) \quad (4.6)$$

that is, the entropy of an air parcel can be written as

$$S = c_p \ln[Tp^{-R/c_p}] + S_0 \quad (4.7)$$

where S_0 is some arbitrary constant.

Let us define another related quantity, which is commonly used in meteorology. From definition (4.4) we see that entropy is conserved ($dS=0$) when there is no exchange of heat between the air parcel and the environment ($\delta q = 0$). The processes that don't exchange heat with the environment are called *adiabatic* processes. From entropy in the form (4.7), we have that for adiabatic process the quantity Tp^{-R/c_p} is also conserved. If we bring that parcel's pressure *adiabatically* to some reference pressure p_0 , say, 1000 hPa (mb), then new temperature of the parcel, denoted as θ , will satisfy the equation $Tp^{-R/c_p} = \theta p_0^{-R/c_p}$. This hypothetical temperature θ is called *potential temperature*, and is defined as

$$\theta = T \left[\frac{p_0}{p} \right]^{R/c_p} \quad (4.8)$$

Thus, the potential temperature is the temperature that an air parcel would have if its pressure is brought *adiabatically* to 1000 mb. Using potential temperature, the entropy is defined as

$$S = c_p \ln \theta \quad (4.9)$$

For adiabatic air parcel, the potential temperature (and, hence, the entropy) is always conserved or “frozen”, regardless where the parcel goes. For example, if such a parcel is rising, its pressure will drop as the atmospheric pressure decreases with height. According to (4.8), the “ordinary” or *kinetic* temperature is given by

$$T = \theta \left[\frac{p}{p_0} \right]^{R/c_p} \quad (4.10)$$

For the given potential temperature θ , the air temperature will decrease with decreasing pressure. This is easy to understand if one realizes that reduction of environmental pressure would make the air expand, which means that the work would have to be done by the air parcel against the environment, which can be done only at the expense of parcel's internal energy, that is, by cooling. Conversely, for a descending parcel, the pressure would increase, at the parcel's temperature would rise due to the work done by the environment while compressing the air parcel.