

11. Diffusional Growth of Cloud Droplets

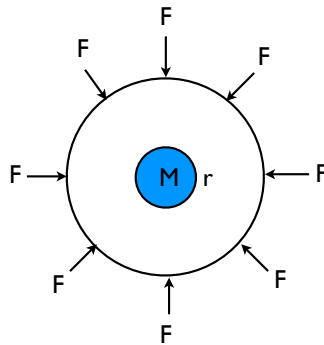
Let us now consider a process of condensation/evaporation of water vapor on cloud droplets. Physically, considering small size of cloud droplets, the only mechanism by which a droplet exchanges moisture and heat with its environment is the molecular diffusion. We will make the following assumptions:

- The diffusion of moisture and heat conduction through the air is steady, that is there is a constant-in-time flux of moisture or heat to or from the droplet;
- The relative humidity right near the droplet's surface is exactly 100%;
- Droplets are far enough from each other, so they don't affect each other's growth;

The magnitude of the diffusive flux of any substance including water vapor in the air is proportional to the magnitude of the *gradient* of that substance, that is

$$F = D \left| \frac{d\rho_v}{dx} \right| \quad (11.1)$$

where the coefficient of proportionality D is called diffusion coefficient with units of m^2/s . For diffusion of water vapor in air, for normal conditions, $D = 2.2 \times 10^{-5} \text{ m}^2/\text{s}$. The heat conduction flux can be written similarly. Note that the units of the flux (11.1) are $\text{kg}_{\text{H}_2\text{O}}/\text{m}^2/\text{s}$, which is consistent with the definition of the flux as something-per-unit-area-per-unit-time. Let's surround our droplet of radius r with the imaginary sphere of a greater radius x as shown in the figure below:



From symmetry, we can safely assume that the flux towards the droplet is the same everywhere on the surface of that sphere. The flux (1.11) is per unit area; therefore, in order to compute the total flux F_{total} of water vapor towards the droplets, we need to multiply (11.1) by the total area of the imaginary sphere $4\pi x^2$, that is

$$F_{total} = 4\pi x^2 F = 4\pi x^2 D \frac{d\rho_v}{dx} \quad (11.2)$$

Now, does this total flux of water vapor depend on the radius of our imaginary sphere? The answer is no, because we assumed that diffusion is steady, and therefore the moisture would not accumulate somewhere around the droplet. The units of F_{total} are $m^2(\text{area}) \times \text{kg}_{\text{H}_2\text{O}}/m^2/s(\text{flux}) = \text{kg}_{\text{H}_2\text{O}}/s$. See, there are no units of length or area here, so the total flux indeed does not depend of radius of the imaginary sphere. In this sense, the total flux is not a flux, per se; rather, is just a rate of change of mass of our droplet M :

$$\frac{dM}{dt} = F_{total} = \text{const}$$

The rate can be also written in terms of the rate of change of droplet's radius:

$$\frac{dM}{dt} = 4\pi\rho_L r^2 \frac{dr}{dt} \quad (11.3)$$

The right-hand-side of (11.2) is constant, which means that $x^2 \frac{d\rho_v}{dx}$ is also constant,

which implies that $\rho_v(x) = \frac{A}{x} + B$, where A and B are some constants. We can find these constant using boundary conditions. Very far away from the droplet, that is when $x \rightarrow \infty$, the term with A drops out and, hence, $\rho_v(\infty) = B$. At the droplet surface, $x = r$, $\rho_v(r) = \frac{A}{r} + \rho_v(\infty)$, that is $A = -r[\rho_v(\infty) - \rho_v(r)]$; therefore

$$\rho_v(x) = \rho_v(\infty) - \frac{r}{x}[\rho_v(\infty) - \rho_v(r)]$$

Differentiating the above expression with respect to x , we find

$$\frac{d\rho_v}{dx} = \frac{r}{x^2}[\rho_v(\infty) - \rho_v(r)] \quad (11.4)$$

Substituting (11.4) to (11.2), we obtain that

$$F_{total} = 4\pi D r [\rho_v(\infty) - \rho_v(r)] \quad (11.5)$$

Using the equality of total flux and the rate of change of droplet's mass (11.3), we can get that

$$\frac{dr}{dt} = \frac{D}{\rho_L r} [\rho_v(\infty) - \rho_v(r)] \quad (11.6)$$

An assumption that the air right near the droplet's surface is saturated is quite reasonable. But at what temperature? The latent heat of condensation would heat the droplet above the temperature of the environment, and vice versa; therefore, there will be a flux of heat away from the droplet. In quasi-equilibrium, the amount of the latent heating due to moisture diffusion flux should balance the heat flux or the temperature diffusion flux. The derivation of (11.5) can be applied to the temperature diffusion problem, except that the diffusion coefficient D should be replaced with the thermal conductivity coefficient K . For normal conditions, $K=2.04 \times 10^{-2} \text{ J m}^{-1} \text{ s}^{-1} \text{ K}^{-1}$. The balance of heat can then be written as:

$$LF_{total} = 4\pi LDr[\rho_v(\infty) - \rho_v(r)] = -4\pi Kr[T(\infty) - T(r)] \quad (11.7)$$

where minus sign at r.h.s. is due to the fact that the flux of temperature is always opposite in direction to the flux of vapor. We still don't know the value for $\rho_v(r) = \rho_{vs}(T(r))$ as we don't know $T(r)$, but in reality we don't really need to know it as we can assume that the heat conduction is as efficient as the vapor diffusion, so that the temperature of a droplet is not very different from the temperature of the environment. Then, we can linearize the dependence of saturation vapor density with respect to temperature:

$$\rho_v(r) = \rho_{vs}(T(r)) = \rho_{vs}(T(\infty)) + \frac{\partial \rho_{vs}}{\partial T} [T(r) - T(\infty)] \quad (11.8)$$

From (11.7)

$$T(r) - T(\infty) = \frac{LD}{K} [\rho_v(\infty) - \rho_v(r)]$$

Substituting this expression to (11.8), after some manipulation, we can arrive to the following expression

$$\rho_v(\infty) - \rho_v(r) = \frac{\rho_v(\infty) - \rho_{vs}(T(\infty))}{1 + \frac{LD}{K} \frac{\partial \rho_{vs}}{\partial T}}$$

Indeed, we don't need to know the temperature of the droplet! Substituting this expression into (11.6), and using the Clausius-Clapeyron equation in the form

$$\frac{\partial \rho_{vs}}{\partial T} \approx \frac{L\rho_{vs}}{R_v T^2},$$

we get

$$\frac{dr}{dt} = \frac{D}{\rho_L \left[1 + \frac{DL^2 \rho_{vs}}{R_v KT^2} \right] r} [\rho_v(\infty) - \rho_{vs}(T(\infty))]$$

Dividing and multiplying by $\rho_{vs}(T(\infty) = \rho_{vs}$ and using definition of relative humidity

$R_h = \frac{e}{e_s} = \frac{\rho_v}{\rho_{vs}}$, we can obtain the following expression for the rate of growth of radius

of a droplet by the diffusion of water vapor:

$$\boxed{\frac{dr}{dt} = \frac{D\rho_{vs}}{r\rho_L \left(1 + \frac{DL^2 \rho_{vs}}{R_v KT^2} \right)} (R_h - 1)} \quad (11.9)$$

Thus, the rate of change of a cloud droplet's radius due to condensation ($R_h > 1$) or evaporation ($R_h < 1$) is inversely proportional to the droplet's radius, that is larger droplets grow relatively slower than smaller droplets. As the consequence, smaller droplets during diffusional growth are catching up with larger droplets, so that the size difference among cloud droplets tends to diminish with time. In other words, the diffusional growth of cloud droplets tends to make them all be of similar size.

The value of the second terms in the parentheses in the denominator of (11.9) depends strongly on temperature because of ρ_{vs} . For instance, it's value is about 2.4 at +20°C, 1.2 at +10°C, 0.6 at 0°C, and 0.3 at -10°C. The air in the clouds is typically colder than 10°C, so in theoretical analysis this terms is often ignored, and a simpler formula is used

$$\frac{dr}{dt} = \frac{D\rho_{vs}}{r\rho_L} (R_h - 1) \quad (11.10)$$

We can find the time t it would take for a droplet to grow (or evaporate depending on R_h) from some initial radius r_o to r assuming that the vapor density and relative humidity is constant by integrating (11.10) as follows

$$\int_{r_o}^r r dr = \frac{D\rho_{vs}}{\rho_L} (R_h - 1) \int_0^t dt$$

so that

$$t = \frac{(r^2 - r_o^2)\rho_L}{2D\rho_{vs}(R_h - 1)}$$