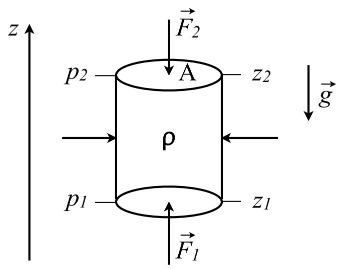
2. Hydrostatic balance equation

Let's consider a balance of forces acting on an imaginary cylinder of air (no solid walls) in the atmosphere as shown on the figure below. Let's assume that the cylinder and the surrounding atmosphere are at rest. In atmospheric physics, it is customary to denote vertical coordinate (height) by z, and horizontal coordinates by x and y. Let the vertical coordinate axis point up. As our cylinder of air is at rest, there is no net force acting on it. The only horizontal forces acting on the cylinder are due to the air pressure (although the pressure itself is not a force), and those pressure forces cancel each other (otherwise the cylinder would accelerate sideways, but the fluid is at rest). In the vertical, for the cylinder to be at rest, the pressure forces $\vec{F}_1$ and $\vec{F}_2$ acting on both bases of the cylinder must balance the weight of the cylinder $m\vec{g} = \rho V\vec{g} = \rho A(z_2 - z_1)\vec{g}$, where A is the cylinder's base area, $z_2 - z_1$ is its height, and g (=9.8 m/s^2) is the acceleration of gravity.



The balance of forces in the vertical is then

$$\vec{F}_1 + \vec{F}_2 + \rho A(z_2 - z_1)\vec{g} = 0$$

Projection of this vector equation on the vertical direction yields scalar equation

$$F_1 - F_2 - \rho A(z_2 - z_1)g = 0$$

Again, the pressure itself is not a force. It becomes one when multiplied by the area it is acting on, that is, $F_1 = Ap_1$ and $F_2 = Ap_2$, so that the above equation can be rewritten as

$$Ap_1 - Ap_2 - \rho A(z_2 - z_1)g = 0$$

After cancelling the area A from the both sides, we can write

$$\frac{p_2 - p_1}{z_2 - z_1} = \frac{\Delta p}{\Delta z} = -\rho g$$

In the limit of a very small Δ, the ratio becomes a derivative (remember definition of the derivative?), and the hydrostatic balance equation becomes

$$\boxed{\frac{dp}{dz} = -\rho g} \qquad (2.1)$$

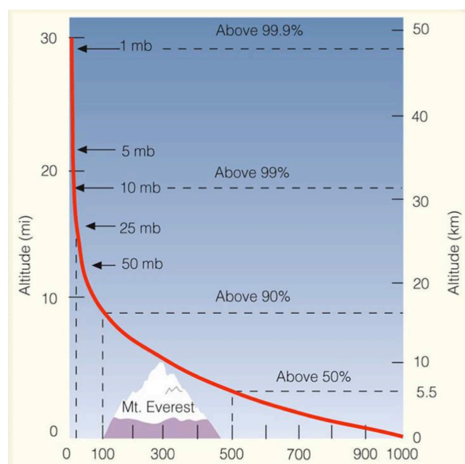
The surface pressure p_0 is found by integrating (2.1) over the entire depth of atmosphere

$$p_0 = \int_0^{p_0} dp = -\int_\infty^0 \rho g\,dz = \int_0^\infty \rho g\,dz = g\int_0^\infty \rho\,dz = gM_a$$

where M_a is the mass of atmosphere per unit area. Thus, the surface pressure is simply the weight of atmospheric column over unit area. As the surface pressure in SI unites is about 10^5 *Pa*, and *g* is about 10 m/s^2, the total mass of atmosphere above a unit (1 m^2) area is $M_a = p_0/g$, which is about 10,000 *kg*, or 10 metric tons. The whole area of the Earth is $4\pi R_e^2$ =4 x 3.14 x 6,360,000^2 m^2 = 5x10^{14} m^2, so the total mass of the atmosphere is about 5x10^{15} metric tons. The same mass would be of a cube of water with the edge of 106 miles. It is kind of neat that we can measure the mass of the entire atmosphere just by measuring the surface pressure by a simple barometer!

Let's express the pressure at some height z through the surface pressure. The hydrostatic equation (2.1) can be rewritten as

$$dp = -\rho g dz$$



Using the equation of state (1.2), it becomes

$$dp = -\frac{p}{RT} g dz$$

or

$$\frac{dp}{p} = -\frac{g}{RT} dz \qquad (2.2)$$

The temperature of the Earth atmosphere changes in relatively narrow range, from about 200K at the tropopause to 300K at the surface; therefore, we can approximate the temperature in the atmosphere as being constant, or *isothermal*, say, at T=250 K. Integrating (2.2) from the surface pressure to some pressure p, and from corresponding surface height 0 to the height z

$$\int_{p_0}^{p} \frac{dp}{p} = \ln p\Big|_{p_0}^{p} = \ln\frac{p}{p_0} = -\int_0^z \frac{g}{RT} dz = -\frac{g}{RT}\int_0^z dz = -\frac{gz}{RT}$$

or we get the following equation for the dependence of pressure on height

$$p = p_0 e^{-\frac{gz}{RT}} \tag{2.3}$$

where p_0 is the surface pressure. Thus, in the isothermal atmosphere, pressure falls exponentially with height. In other words, pressure falls by a factor of e=2.718 over the depth $H = \frac{RT}{g}$, which is about 7.2 km. Note that in the Earth atmosphere, the deviation of pressure from (2.3) is indeed very small. The dependence of pressure on altitude is illustrated by the figure above. One can see that indeed the pressure falls very quickly with height. Remember, the pressure at any arbitrary height above the surface would also equal the weight of atmosphere above that height. The equation (2.3) would still apply to pressure at levels above that new reference level with reference pressure p_0, and with z being the height above that new reference level.