12. Growth of Cloud Droplets in a Steady Updraft

Consider a parcel rising above the Lifting Condensation Level (LCL) with constant vertical velocity w. Let's derive an equation that describes the rate of change of supersaturation $S = R_h - 1$. In general, the rate of change of S can be written as

$$\frac{dS}{dt} = P - C \qquad (12.1)$$

where P is production by adiabatic cooling due to parcel rise, a C is reduction due to growth of cloud droplets, or condensation. First, let's derive the expression for P. For moist-adiabatic ascent, the production rate can be derived as

$$P = \left[\frac{dS}{dt}\right]_{ad} = \left[\frac{dR_h}{dt}\right]_{ad} = \left[\frac{d}{dt}\frac{e}{e_s}\right]_{ad} = -\frac{e}{e_s^2}\left[\frac{de_s}{dt}\right]_{ad} \approx \left[\frac{d\ln e_s}{dt}\right]_{ad} = \frac{d\ln e_s}{dT}\left[\frac{dT}{dt}\right]_{ad} = -\frac{\Gamma_m L}{R_v T^2}w$$

Above, we used the fact that in cloud $R_h \approx 1$, e is approximately conserved, the parcel is cooling at moist-adiabatic rate (see 8.4), and also employed differential form of the Clausius-Clapeyron equation:

$$\frac{d\ln e_s}{dT} = \frac{L}{R_v T^2}$$

Note that in the case of non-cloudy updraft, Γ_m is replaced with dry-adiabatic rate $\Gamma_a = -g/c_p$. The production rate of supersaturation is proportional to vertical velocity. We can estimate that production of supersaturation in units of %/s to be about $P[\%/s] \approx 0.04w$ (using T=273K and Γ_m=0.005 K/m). For instance, for 5 m/s updraft, a cloudy air would increase relative humidity at the rate of about 0.2% per second. However, in a cloudy updraft, condensation counteracts this tendency by removing vapor from the air by converting it into the liquid water in growing cloud droplets. The rate of change of supersaturation due to condensation is derived as

$$C = \left[\frac{dS}{dt}\right]_C = \left[\frac{dR_h}{dt}\right]_C = \left[\frac{d}{dt}\frac{q_v}{q_s}\right]_C \approx \frac{1}{q_s}\left[\frac{dq_v}{dt}\right]_C = \frac{\rho}{\rho_{vs}}\left[\frac{dq_v}{dt}\right]_C$$

The total amount of water (vapor q_v plus cloud water q_c) in a parcel is assumed to be constant, that is

$$\left[\frac{dq_v}{dt}\right]_C = -\left[\frac{dq_c}{dt}\right]_C$$

Let's now assume that all drops have same average radius r_c and concentration N_c, so the liquid water mixing ratio is

$$q_c = \frac{\rho_c}{\rho} = \frac{4}{3}\frac{\pi r_c^3 \rho_L N_c}{\rho}$$

The assumption that all droplets are about the same size is not a bad one. As mentioned in the previous chapter, smaller droplets tend to grow faster than the bigger ones, so, in relatively short time, the droplets' sizes tend to catch up to each other. The rate of change of cloud water is given by

$$\left[\frac{dq_c}{dt}\right]_C = \frac{4\pi r_c^2 \rho_L N_c}{\rho}\left[\frac{dr_c}{dt}\right]_C$$

Here we assumed that CCN activation is over, that is N_c is constant. Using the expression for droplet growth rate (11.10), C can be expressed as

$$C = \frac{4\pi r_c^2 \rho_L N_c}{\rho_{vs}}\left[\frac{dr_c}{dt}\right]_C = 4\pi D r_c N_c S$$

or $C = \frac{S}{\tau_p}$, where $\tau_p = \frac{1}{4\pi D r_c N_c}$ with units of time. This time scale is called *phase relaxation time*. The physical meaning of τ_p can be better understood if we consider a cloudy air parcel at rest, so that there is no production of supersaturation due to lifting. Let's assume that due to some perturbation, supersaturation S_o has suddenly developed. It would immediately start decreasing due to droplets' growth with the rate (P=0 here)

$$\frac{dS}{dt} = -\frac{S}{\tau_p}$$

This differential equation can easily be solved

$$S(t) = S_o e^{-\frac{t}{\tau_p}}$$

This is an exponential decay equation. One can see that initial supersaturation S_o would decrease by a factor of e=2.718… over time period equal to τ_p, which is also called e-folding time. Over time period of $2\tau_p$, supersaturation would decrease by a factor of 7.4, or close to an order of magnitude. Thus, τ_p represents the amount of time it take for the supersaturation to be effectively eliminated in absence of supersaturation production. The value of τ_p can be estimated assuming typical value of droplet radius in cloud about 10 μm = 10^{-5} m, and typical concentration of about

500 cm^{-3}, so τ_p=0.7 s. So, the typical time over which supersaturation is eliminated in warm (liquid water) clouds is just a few seconds!

In a steady updraft, the supersaturation is quickly approaching the equilibrium value S_{eq} achieved when production by lifting exactly balances destruction by condensation, that is $P = C$, so that

$$S_{eq} = -\frac{\Gamma_m L}{R_v T^2} \tau_p w$$

For T=273K, Γ_m=0.005 K/m, τ_p=0.7 s, and w=5 m/s, S_{eq}=0.2%.

So far, it was assumed that cloud drop concentration is constant, that is, the cloudy updraft is above the region near the cloud base where CCN are activated. The maximum cloud droplet concentration in the updraft equals the number of activated CCN. The later is determined by the maximum supersaturation reached at the cloud base. Right above the level where the very first CCN (droplets of haze) start activating, the condensation rate is relatively small compared to the production rate; therefore, the supersaturation would keep increasing. As more and more droplets of haze become cloud droplets, the condensation rate increases, which would slow down the rate of supersaturation increase. Eventually, the supersaturation would reach a maximum.

The differential equation that describes the rate of increase of supersaturation in the CCN activation region is rather complex and cannot be generally solved analytically; however, it is possible to find an approximate solution like the one proposed by Twomey (1959) for a specific case when CCN activation spectrum is assumed to be a power-law

$$N_a = CS^k \tag{12.2}$$

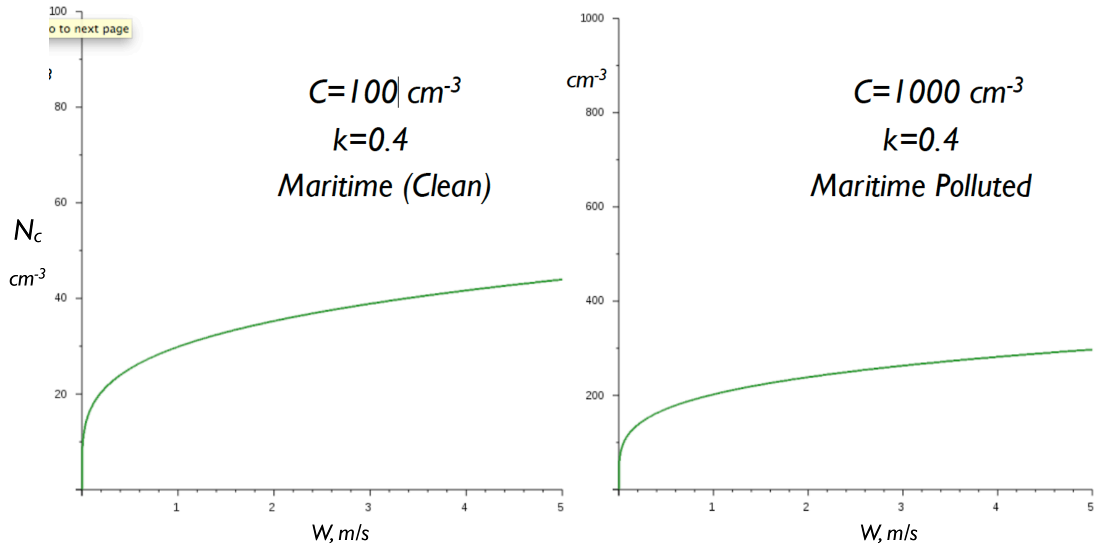
where N_a is the number of activated CCN at supersaturation S (in %), while C (in cm^{-3}) and k are empirical parameters that depend on air-mass, etc. Then, the concentration of cloud droplets in the updraft with vertical velocity w (in cm/s) is approximated by

$$N_c = 0.88 C^{\frac{2}{k+2}} \left[0.07 w^{3/2}\right]^{\frac{k}{k+2}} \tag{12.3}$$

with corresponding maximum supersaturation (in %) computed as

$$S_{\max} = (N_c / C)^{\frac{1}{k}} = 0.88^{\frac{1}{k}} C^{-\frac{1}{k+2}} \left[0.07 w^{3/2}\right]^{\frac{1}{k+2}}$$

Figure below demonstrates the dependence of cloud drop concentration on vertical velocity when using (12.3) for two different air-masses, maritime clean and polluted. One can see, that for same 5 m/s updraft, the cloud drop concentration for clean airmass (low CCN count) and polluted airmass can be quite different, that is, about 40 cm^{-3} for clean and 400 cm^{-3} for polluted.



The vertical profiles of supersaturation and cloud droplet concentration above visible cloud base (LCL) are sketched below. The peak of supersaturation at the cloud base determines the cloud droplet concentration. Typically, that peak is located about 10-20 m above the visible cloud base. Above it, the supersaturation tends to quickly decrease toward the equilibrium value, and concentration of cloud droplets remains constant.