Boussinesq Approximation

The atmospheric boundary layer is relatively shallow, so it is tempting to ignore variation of air density with height. Besides, there is a great wealth of methods and results that have been obtained over more than a century of intensive work by various scientists in both theory and laboratory studies that deal with nearly incompressible fluids like water when the density variations are indeed can be ignored. In ABL studies, it is common to make intensive use of the so-called Boussinesq approximation, which was proposed by Joseph Boussinesq, a French mathematician, who lived in the late 19th century. Basically the idea behind the approximation is to assume that the air is indeed incompressible except for the buoyancy effects when small variations of density need to be taken into account. In addition, we assume that density is constant, that is, it doesn't depend on height. This approximation is especially plausible for studies of the ocean.

If we logarithmically differenciate the ideal gas equation, assume the hydrostatic balance, and replace the differencials with the differences, we get the following equation for the relative change of density with height:

$$\frac{\Delta\rho}{\rho} = -\frac{g\Delta z}{RT} - \frac{\Delta T}{T} \quad (1)$$

So, for typical boundary layer height of 1000 m, temperature 300 K, and temperature change of 10 K, we get that the variation of density, mostly because of the first term in the r.h.s. of (1), would be about 10%. Thus, our assumption of constant-with-height density is ABL is justified. The continuity equation in Eulerian form (rate of change in a point) is written as

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{u} = 0 \quad (2)$$

It can be rewritten in equivalent, Lagrangian, form (following air parcel's trajectory) as

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla\cdot\mathbf{u} = 0 \quad (3)$$

so, the assumption of constant density along the parcel trajectory results in the 'non-divergence' equation for incompressible fluid:

$$\nabla\cdot\mathbf{u} = 0 \quad (4)$$

The inviscid momentum equation (Second Newton's Law) can be written as

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla p + \rho\mathbf{g} \quad (5)$$

If the fluid is at rest, its weight is balanced by the pressure-gradient force. Typically, in the ABL, acceleration of fluid is much smaller than acceleration of gravity; therefore, the net force is just a small deviation from forces maintaining the hydrostatic balance (one of which is gravity). It is small fluctuations of density or *buoyancy* that are responsible for the unbalance.

Let's partition the pressure and density into basic hydrostatically balanced state and perturbations from that state:

$p(x,y,z,t) = p_0(z) + p'(x,y,z,t)$

$\rho(x,y,z,t) = \rho_0 + \rho'(x,y,z,t)$.

Let's also assume that the perturbations are small compared to the basic state values. Substituting the above expressions into (5), using hydrostatic equation $-\nabla p_o + \rho_o \mathbf{g} = 0$, ignoring the effect of density fluctuations on the momentum itself, and ignoring second-order perturbation terms, we can rewrite (5) as

$$\rho_o \frac{d\mathbf{u}}{dt} = -\nabla p' + \rho' \mathbf{g} \qquad (6)$$

Dividing both sides by a constant density, we get

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho_o}\nabla p' + \frac{\rho'}{\rho_o}\mathbf{g} \qquad (7)$$

Differentiating logarithmically the ideal gas law, and replacing differentials with the perturbations, we can write

$$\frac{p'}{p_o} = \frac{\rho'}{\rho_o} + \frac{T'}{T_o} \qquad (8)$$

It can be shown that relative pressure perturbations are much smaller than relative temperature perturbations (which can be shown to be relatively small for flow speeds much lower than the speed of sound); therefore,

$$\frac{\rho'}{\rho_o} \approx -\frac{T'}{T_o} \qquad (9)$$

Temperature is not conservative quantity following air parcel's adiabatic path, but potential temperature is. Therefore, it is convenient to replace temperature perturbations with potential temperature perturbations. Differentiating logarithmically the definition of the potential temperature, and replacing differential with perturbation, one can get

$$\frac{\theta'}{\theta_o} = \frac{T'}{T_o} - \frac{R}{c_p}\frac{p'}{p_o} \qquad (10)$$

Ignoring the relative pressure perturbations, (9) can be written as

$$\frac{\rho'}{\rho} \approx -\frac{\theta'}{\Theta} \qquad (11)$$

where we ignored the dependence of background temperature on height and substituted it with some mean temperature Θ. The Boussinesq momentum equation is then written as

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho_o}\nabla p' - \frac{\theta'}{\Theta}\mathbf{g} \qquad (12)$$

The potential perturbations are found from conservation of potential temperature:

$$\frac{d\theta}{dt} = 0 \qquad (13)$$

Substituting potential temperature with the sum of background and perturbation: $\theta = \theta_o(z) + \theta'$, the equation (13) can be written as

$$\frac{d\theta'}{dt} + w\frac{d\theta_o}{dz} = 0 \qquad (13)$$

Summarizing, the set of equations in Boussinesq approximation for the inviscid fluid in component form can be written as

Momentum equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial \pi}{\partial x} \qquad (14)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial \pi}{\partial y} \qquad (15)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial \pi}{\partial z} + \beta\theta' \qquad (16)$$

Continuity equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (17)$$

Thermodynamics equation:

$$\frac{\partial \theta'}{\partial t} + u\frac{\partial \theta'}{\partial x} + v\frac{\partial \theta'}{\partial y} + w\frac{\partial \theta'}{\partial z} = -w\frac{\partial \theta_o}{\partial z} \qquad (18)$$

Here, we used the expansion of the material derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$, introduced a new pressure variable $\pi = p'/\rho_o$ and a constant $\beta = g/\Theta$.