11. Diffusional Growth of Cloud Droplets

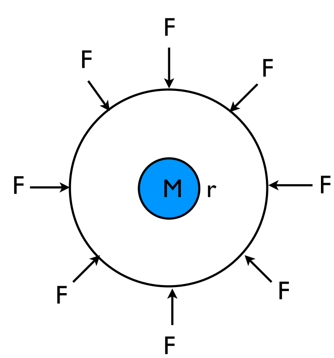
Let us now consider a process of condensation or evaporation of water vapor on a cloud droplet. Physically, considering small size of the cloud droplets, the only mechanism by which a droplet exchanges moisture and heat with the environment is the molecular diffusion. The following assumption are made:

- The diffusion is steady, that is there is a constant or steady flux of moisture towards the droplet from afar;
- There is exactly 100% relative humidity right near the droplet's surface at all times, that is we assume that very close to the droplet surface, the vapor is at equilibrium with liquid water;
- Droplets are far from each other, so they don't affect each other and grow independently;
- Heat exchange with the environment is very efficient, so that the temperature of a droplet is the same as that of the environment.

The magnitude of the diffusion flux of any substance including water vapor density in air can be written as proportional to the magnitude of the *gradient* of that substance, or how fast it's concentration changes with the distance, that is

$$F = D\left|\frac{d\rho_v}{dx}\right| \tag{11.1}$$

where the coefficient of proportionality D is called diffusion coefficient with units of m^2/s. For diffusion of water vapor in air, for normal conditions, $D = 2.2\text{x}10^{-5}$ m^2/s. Note that the units of the flux (11.1) are $kg_{H2O}/m^2/s$, which is consistent with the definition of the flux as something-per-unit-area-per-unit-time. Let's surround our droplet of radius r with the imaginary sphere of a greater radius x as shown in the figure below:



From symmetry, we can safely assume that the flux towards the droplet is the same everywhere on the surface of that sphere. The flux (1.11) is per unit area; therefore, in order to compute the total flux F_{total} of water vapor towards the droplets, we need to multiply (11.1) by the total area of the imaginary sphere $4\pi x^2$, that is

$$F_{total} = 4\pi x^2 F = 4\pi x^2 D \frac{d\rho_v}{dx} \qquad (11.2)$$

Now, does this total flux of water vapor depend on the radius of our imaginary sphere? The answer is no, because we assumed that diffusion is steady, which means that all the vapor is coming from very far from the droplet; otherwise, it would accumulate somewhere around the droplet, which would violate steady-state assumption. The units of F_{total} are m^2(area)x kg$_{H2O}$/m^2/s(flux)= kg$_{H2O}$/s. See, there are no units of length or area here, so the total flux indeed does not depend of radius of the imaginary sphere. In this sense, the total flux is not a flux, per se; rather, is just a rate of change of mass of our droplet M :

$$\frac{dM}{dt} = F_{total} = \text{constant}$$

The rate can be also written in terms of the rate of change of droplet's radius:

$$\frac{dM}{dt} = \frac{d}{dt}(\frac{4}{3}\pi r^3 \rho_L) = \frac{4}{3}\pi\rho_L \frac{dr^3}{dt} = \frac{4}{3}\pi\rho_L \frac{dr^3}{dr}\frac{dr}{dt} = 4\pi\rho_L r^2 \frac{dr}{dt} \qquad (11.3)$$

The right-hand-side of (11.2) is constant, which means that $x^2 \frac{d\rho_v}{dx}$ is constant, which means that $\rho_v(x) = \frac{A}{x} + B$, where A and B are some constants. We can find these constant using boundary conditions. Very far away from the droplet, that is when $x \to \infty$, the term with A drops out and, hence, $\rho_v(\infty) = B$. At the droplet surface, $x = r$, $\rho_v(r) = \frac{A}{r} + \rho_v(\infty)$, that is $A = -r[\rho_v(\infty) - \rho_v(r)]$; therefore

$$\rho_v(x) = \rho_v(\infty) - \frac{r}{x}[\rho_v(\infty) - \rho_v(r)]$$

Differentiating the above expression with respect to x, we find

$$\frac{d\rho_v}{dx} = \frac{r}{x^2}[\rho_v(\infty) - \rho_v(r)] \qquad (11.4)$$

Substituting (11.4) to (11.2), we obtain that

$$F_{total} = 4\pi Dr[\rho_v(\infty) - \rho_v(r)] \qquad (11.5)$$

Using the equality of total flux and the rate of change of droplet's mass (11.3), we can get that

$$\frac{dr}{dt} = \frac{D}{\rho_L r}[\rho_v(\infty) - \rho_v(r)] \qquad (11.6)$$

A reasonable assumption is that the air right near the droplet's surface is saturated. But at what temperature? The latent heat of condensation would heat the droplet above the temperature of the environment, and vise versa; therefore, there will be the flux of heat away from the droplet. In quasi-equilibrium, the amount of the latent heating due to moisture diffusion flux should balance the heat flux or the temperature diffusion flux. The derivation of (11.5) can be applied to the temperature diffusion problem, except that the diffusion coefficient *D* should be replaced with the thermal conductivity coefficient *K*. For normal conditions, K=2.04x10^{-2} J m^{-1} s^{-1} K^{-1}. The balance of heat can then written as:

$$LF_{total} = 4\pi LDr[\rho_v(\infty) - \rho_v(r)] = -4\pi Kr[T(\infty) - T(r)] \qquad (11.7)$$

where the minus sign at r.h.s. is due to the fact that the flux of temperature is always opposite in direction to the flux of vapor. We still don't know the value for $\rho_v(r) = \rho_{vs}(T(r))$ as we don't know $T(r)$, but in reality we don't really need to know if we assume that the heat diffusion as efficient as the vapor diffusion, so that the temperature of a droplet is not very different from the temperature of the environment. Then, we can linearize the dependence of saturation vapor density with respect to temperature:

$$\rho_v(r) = \rho_{vs}(T(r)) = \rho_{vs}(T(\infty)) + \frac{\partial \rho_{vs}}{\partial T}[T(r) - T(\infty)] \qquad (11.8)$$

From (11.7)

$$T(r) - T(\infty) = \frac{LD}{K}[\rho_v(\infty) - \rho_v(r)]$$

Substituting this expression to (11.8), after some manipulation, we can arrive to the following expression

$$\rho_v(\infty) - \rho_v(r) = \frac{\rho_v(\infty) - \rho_{vs}(T(\infty))}{1 + \frac{LD}{K}\frac{\partial \rho_{vs}}{\partial T}}$$

so that we don't need to know the temperature of the droplet! Substituting this expression to (11.6), and using the Clausius-Clapeyron equation in the form $\frac{\partial \rho_{vs}}{\partial T} \approx \frac{L\rho_{vs}}{R_v T^2}$, we get

$$\frac{dr}{dt} = \frac{D}{\rho_L \left[1 + \frac{DL^2 \rho_{vs}}{R_v K T^2}\right] r} [\rho_v(\infty) - \rho_{vs}(T(\infty)]$$

Dividing and multiplying by $\rho_{vs}(T(\infty) = \rho_{vs}$ and using the definition of the relative humidity $R_h = \frac{e}{e_s} = \frac{\rho_v}{\rho_{vs}}$, we finally obtain the expression for the rate of growth of radius of a droplet by the diffusion of water vapor:

$$\boxed{\frac{dr}{dt} = \frac{D\rho_{vs}}{r\rho_L \left(1 + \frac{DL^2 \rho_{vs}}{R_v K T^2}\right)} (R_h - 1)} \tag{11.9}$$

Thus, the rate of change of a cloud droplet's radius due to condensation ($R_h > 1$) or evaporation ($R_h < 1$) is inversely proportional to the droplet's radius, that is larger droplets grow relatively slower than smaller droplets. As the consequence, smaller droplets are catching-up with larger droplets, so that the size difference among cloud droplets diminishes with time. In other words, the diffusional growth of cloud droplets tends to make them all be at about the same size.

The value of the second terms in the parentheses in the denominator of (11.9) depends strongly on temperature because of ρ_{vs}. For example, it's value is about 2.4 at +20^{o}C, 1.2 at +10^{o}C, 0.6 at 0^{o}C, and 0.3 at -10^{o}C. The air in the clouds is typically colder than 10^{o}C, so in theoretical analysis this terms is often ignored, and a simpler formula is used

$$\frac{dr}{dt} = \frac{D\rho_{vs}}{r\rho_L}(R_h - 1) \tag{11.10}$$