## 8. Static Stability of Cloudy Air

Previously, we established that the dry, or more precisely, cloud free air would cool/warm at the rate of about 10 K for every km of adiabatic slow ascent/descent. We also established, that in order a cloud-free layer of air to be statically stable with respect to small vertical displacements, the temperature lapse rate in that layer should be larger than the adiabatic lapse rate $\Gamma_a = -g/c_p$, that is the temperature should decrease with height at the rate smaller than about -10 K per every km of height.

What if the layer is cloudy? The cloudy air is saturated with respect to liquid water (or ice in the case of icy cloud like cirrus). Therefore, lifting air would cool it, but cooling would decrease the saturation vapor pressure, and, as the result, would force condensation, which would warm the parcel. The result is the saturated parcel would cool less when lifted than unsaturated parcel. It could affect the stability criterion that we found for the dry or unsaturated air.

Let's assume that we lift a parcel of cloudy air parcel hydrostatically (slowly) and also adiabatically. By adiabatically we mean that there is no external heat or moisture added to or taken away from that parcel. All the heat would come in the form of internal heat of condensation. The 1st Law of Thermodynamics for such a cloudy parcel can be written as

$$\delta q = Ldq_l = c_p dT - \frac{1}{\rho}dp \qquad (8.1)$$

where $dq_l$ is the increase of liquid water content as the result of condensation. The condensation rate is such that the vapor pressure is always equal to saturation vapor pressure, with all the excess of vapor going to cloud liquid water content. The total water $q_t = q_s + q_l$ should be preserved, that is $dq_t = 0$, or $dq_l = -dq_s$, so (8.1) can be written as

$$-Ldq_s = c_p dT - \frac{1}{\rho}dp \qquad (8.2)$$

Expressing $q_s$ in terms of saturation vapor pressure $q_s = \varepsilon \frac{e_s}{p}$ we have

$$dq_s = d(\varepsilon \frac{e_s}{p}) = \varepsilon \frac{de_s}{p} - \varepsilon \frac{e_s}{p^2}dp = \varepsilon \frac{e_s}{p}\frac{de_s}{e_s} - \varepsilon \frac{e_s}{p}\frac{dp}{p} = q_s \frac{de_s}{e_s} - q_s \frac{dp}{p}$$

Substituting the result into (8.2) and using equation of state, after some rearrangement we get

$$-Lq_s \frac{de_s}{e_s} = c_p dT - \left( \frac{Lq_s}{\rho RT} + \frac{1}{\rho} \right) dp$$

Using the differential form of the Clausius-Clapeyron equation

$$\frac{de_s}{e_s} = \frac{L}{R_V} \frac{dT}{T^2}$$

we get

$$-c_p \left( 1 + \frac{L^2 q_s}{c_p R_V T^2} \right) dT = -\left( \frac{Lq_s}{\rho RT} + \frac{1}{\rho} \right) dp$$

Dividing both sides by $dz$ and using hydrostatic equation $\frac{dp}{dz} = -g\rho$, we have

$$\frac{dT}{dz} = -\frac{g}{c_p} \frac{1 + \frac{Lq_s}{RT}}{1 + \frac{L^2 q_s}{c_p R_V T^2}}$$

Using definition of the dry adiabatic lapse rate and $\varepsilon = R / R_v$, we arrive at the following definition of moist-adiabatic lapse rate:

$$\Gamma_m = \Gamma_a \frac{1 + \frac{Lq_s}{RT}}{1 + \frac{\varepsilon L^2 q_s}{c_p R T^2}} \tag{8.4}$$

It is easy to show that the ratio of nominator to denominator in (8.4) is in the order of 0.1. Also, the denominator is always greater than one; therefore, we conclude that the moist-adiabatic lapse rate is always smaller in magnitude than adiabatic lapse rate:

$$|\Gamma_m| < |\Gamma_a| \tag{8.5}$$

For static stability of the cloudy layer, the temperature lapse rate should decrease slower than moist-adiabatic lapse rate:

$$\frac{dT}{dz} > -|\Gamma_m| \tag{8.6}$$

Unlike constant $\Gamma_a$, $\Gamma_m$ depends strongly on vapor mixing ratio, namely, the magnitude (absolute value) decreases with increasing $q_s$. The latter is the exponential function of temperature, namely, increases rather sharply with temperature, so the moist-adiabatic lapse rate becomes less steep. For example, in Tropics, near the surface, $\Gamma_m \approx$ -3 K/km, which is considerably smaller in magnitude than $\Gamma_a$. At very cold temperatures, specifically, high in troposphere, $q_s$ becomes very small and $\Gamma_m$ approaches the value of $\Gamma_a$.